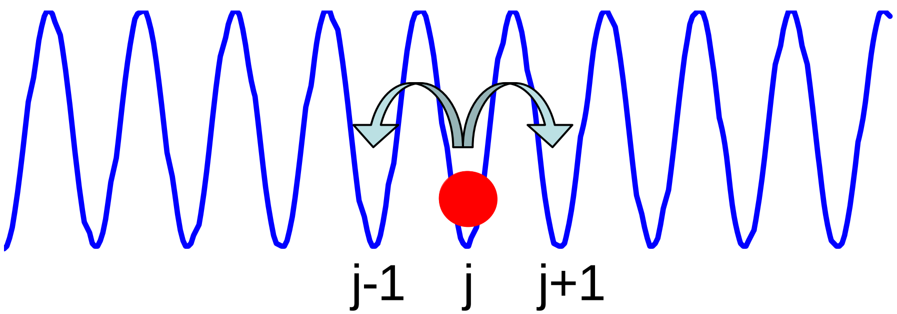
ბოზე-ჰაბარდის მოდელი ორზონიანი სისტემებისთვის და ჰილბერტის სივრცის რიცხვითი მოდელირება MATLAB-ში

Bose-Hubbard Model – one particle

$$\hat{H} = \sum_{j=1}^{N-1} g\left(\hat{a}_{j+1}^{+}\hat{a}_{j} + \hat{a}_{j}^{+}\hat{a}_{j+1} + \hat{b}_{j+1}^{+}\hat{b}_{j} + \hat{b}_{j}^{+}\hat{b}_{j+1}\right) + U\sum_{j=1}^{N}\hat{a}_{j}^{+}\hat{b}_{j}^{+}\hat{a}_{j}\hat{b}_{j}$$

$$\left[\hat{a}_m, \hat{a}_n^{+}\right] = \left[\hat{b}_m, \hat{b}_n^{+}\right] = \delta_{mn} \qquad \left[\hat{a}_m, \hat{a}_n\right] = \left[\hat{b}_m, \hat{b}_n\right] = \left[\hat{a}_m^{\pm}, \hat{b}_n^{\pm}\right] = 0$$



$$|n\rangle = \hat{a}_n^{+}|0\rangle \qquad |\Psi(t)\rangle = \sum_{n=1}^{N} c_n(t)|n\rangle$$

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$$

$$\left(\sum_{j=1}^{N-1}\hat{a}_{j+1}^{+}\hat{a}_{j}\right)\hat{a}_n^{+}|0\rangle = \hat{a}_{n+1}^{+}|0\rangle = |n+1\rangle \qquad \left(\sum_{j=1}^{N}\hat{a}_{j}^{+}\hat{b}_{j}^{+}\hat{a}_{j}\hat{b}_{j}\right)\hat{a}_n^{+}|0\rangle = 0$$

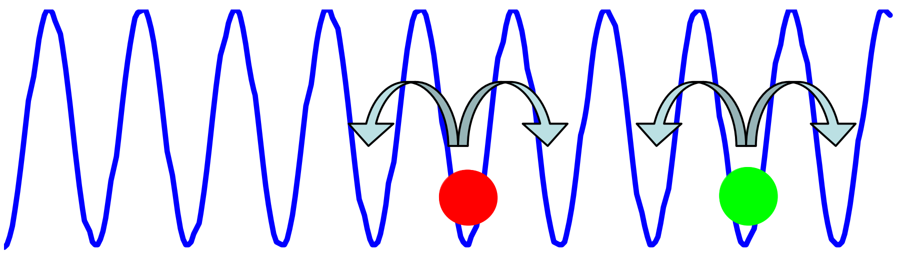
$$\hat{H}|n\rangle = g\left(|n+1\rangle + |n-1\rangle\right)$$

$$i\frac{\partial}{\partial t}c_n(t) = g\left[c_{n+1}(t) + c_{n-1}(t)\right]$$

Bose-Hubbard Model – two particles

$$\hat{H} = \sum_{j=1}^{N-1} g\left(\hat{a}_{j+1}^{+}\hat{a}_{j} + \hat{a}_{j}^{+}\hat{a}_{j+1} + \hat{b}_{j+1}^{+}\hat{b}_{j} + \hat{b}_{j}^{+}\hat{b}_{j+1}\right) + U\sum_{j=1}^{N}\hat{a}_{j}^{+}\hat{b}_{j}^{+}\hat{a}_{j}\hat{b}_{j}$$

$$|m,n\rangle = \hat{a}_{m}^{+}\hat{b}_{n}^{+}|0\rangle$$



$$|\Psi(t)\rangle = \sum_{m,n=1}^{N} c_{mn}(t)|m,n\rangle$$

$$\left(\sum_{j=1}^{N-1}\hat{a}_{j+1}^{+}\hat{a}_{j}\right)\hat{a}_{m}^{+}\hat{b}_{m}^{+}|0\rangle = \hat{a}_{m+1}^{+}\hat{b}_{n}^{+}|0\rangle = |m+1,n\rangle$$

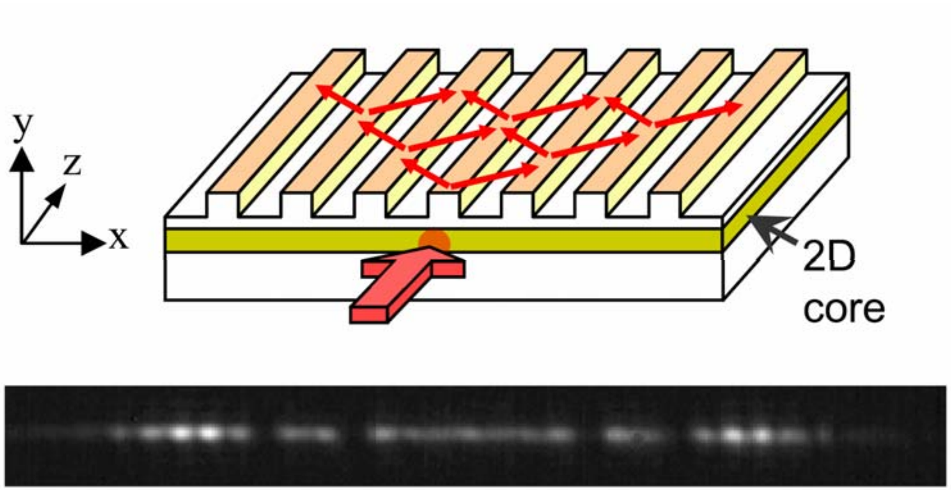
$$\left(\sum_{j=1}^{N}\hat{a}_{j}^{+}\hat{b}_{j}^{+}\hat{a}_{j}\hat{b}_{j}\right)\hat{a}_{m}^{+}b_{n}^{+}|0\rangle = \delta_{mn}\hat{a}_{m}^{+}b_{n}^{+}|0\rangle = \delta_{mn}|m,n\rangle$$

$$\hat{H}|m,n\rangle = g\left(|m,n+1\rangle + |m,n-1\rangle + |m+1,n\rangle + |m-1,n\rangle\right) + \delta_{mn}U|m,n\rangle$$

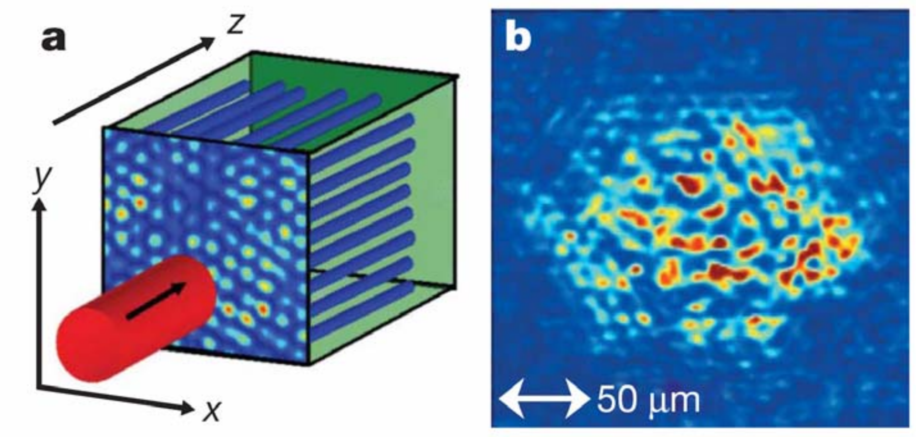
$$i\frac{\partial}{\partial t}c_{mn} = g\left[c_{m,n+1} + c_{m,n-1} + c_{m+1,n} + c_{m-1,n}\right] + \delta_{mn}Uc_{mn}$$

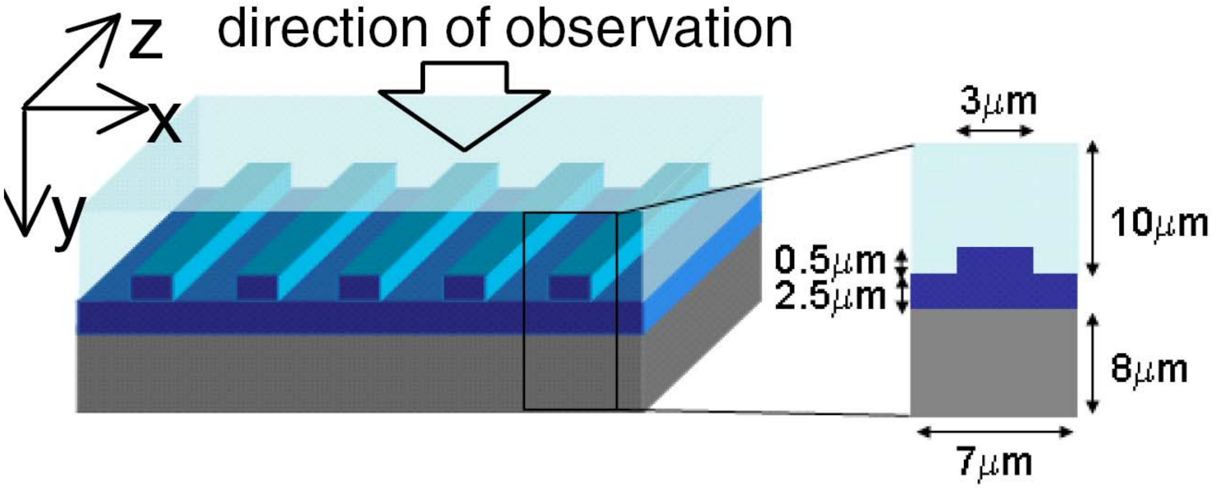
Optical Waveguide Arrays and Lattices

Perets, et.al., PRL **100,** 170506 (2008)



Schwartz, et.al., Nature, **446**, 52 (2007)





Trompeter et.al., PRL **96,** 023901 (2006)

Maxwell Equations

$$\vec{\nabla}\times\vec{E}(\vec{r},t) = -\frac{1}{c}\frac{\partial\vec{B}(\vec{r},t)}{\partial t}; \qquad \vec{\nabla}\times\vec{H}(\vec{r},t) = \frac{1}{c}\frac{\partial\vec{D}(\vec{r},t)}{\partial t}$$

$$\vec{B}(\vec{r},t) = \vec{H}(\vec{r},t) \qquad \Rightarrow \qquad \vec{\nabla}\times\vec{\nabla}\times\vec{E}(\vec{r},t) = -\frac{1}{c^2}\frac{\partial^2\vec{D}(\vec{r},t)}{\partial t^2}$$

$$\vec{E}(\vec{r},t) \equiv E_y(y,z,t)$$

$$\Downarrow$$

$$\frac{\partial^2 E_y(x,z,t)}{\partial x^2} + \frac{\partial^2 E_y(x,z,t)}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 D_y(x,z,t)}{\partial t^2} = 0$$

$$D_y = [n(x)]^2 E_y \qquad n(x) = n_0 + n_0\delta n(x) \qquad D_y = [n_0^2 + 2n_0\delta n(x)]E_y$$

Paraxial Approximation

$$\frac{\partial^2 E_y(x,z,t)}{\partial x^2}+\frac{\partial^2 E_y(x,z,t)}{\partial z^2}-\frac{n_0^2}{c^2}\frac{\partial^2 E_y(x,z,t)}{\partial t^2}-\frac{2n_0\delta n(x)}{c^2}\frac{\partial^2 E_y(x,z,t)}{\partial t^2}=0$$

$$E_y=\Psi e^{-i(\omega t-kz)}+\Psi^* e^{i(\omega t-kz)} \qquad \omega = kc/n_0$$

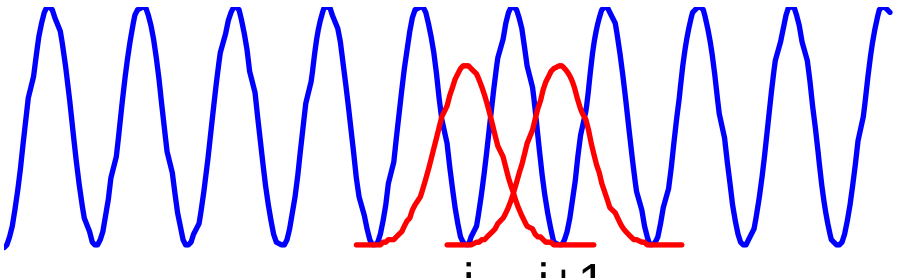
$$E_y=\Psi\left[\varepsilon x,\varepsilon^2 z\right]e^{i(\omega t-kz)}+\left(\Psi\left[\varepsilon x,\varepsilon^2 z\right]\right)^* e^{-i(\omega t-kz)};$$

$$\frac{\partial}{\partial x}\Psi\sim\varepsilon; \qquad \frac{\partial}{\partial z}\Psi\sim\varepsilon^2$$

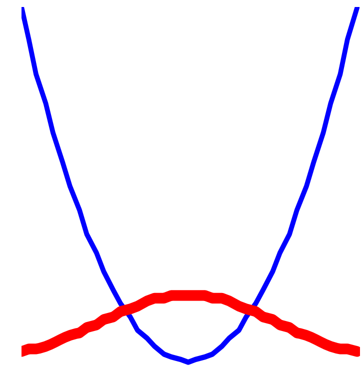
$$2ik\frac{\partial\Psi}{\partial z}+\frac{\partial^2\Psi}{\partial x^2}+\omega^2\frac{2n_0\delta n(x)}{c^2}\Psi=0$$

Tight-Binding Approximation

$$i\frac{\partial \Psi}{\partial z}+\frac{\partial^2 \Psi}{\partial x^2}-V(x)\Psi=0 \qquad V(x)=-\frac{\omega}{c}\delta n(x)$$



$$a\sin^2 x \approx ax^2$$



$$V(x)=ax^2 \qquad \Psi(t,x)=\frac{a^{1/8}}{\pi^{1/4}}e^{-iz\sqrt{a}}e^{-x^2\sqrt{a}/2}$$

$$\Phi_n(x)=\frac{a^{1/8}}{\pi^{1/4}}e^{-(x-n\pi)^2\sqrt{a}/2}$$

$$\Psi(z,x)=e^{-iz\sqrt{a}}\sum_{n=-\infty}^{\infty}\psi_n(z)\Phi_n(x)$$

$$i\frac{\partial \psi_j}{\partial z} + g\left(\psi_{j+1} + \psi_{j-1}\right) + \varepsilon\psi_j = 0$$

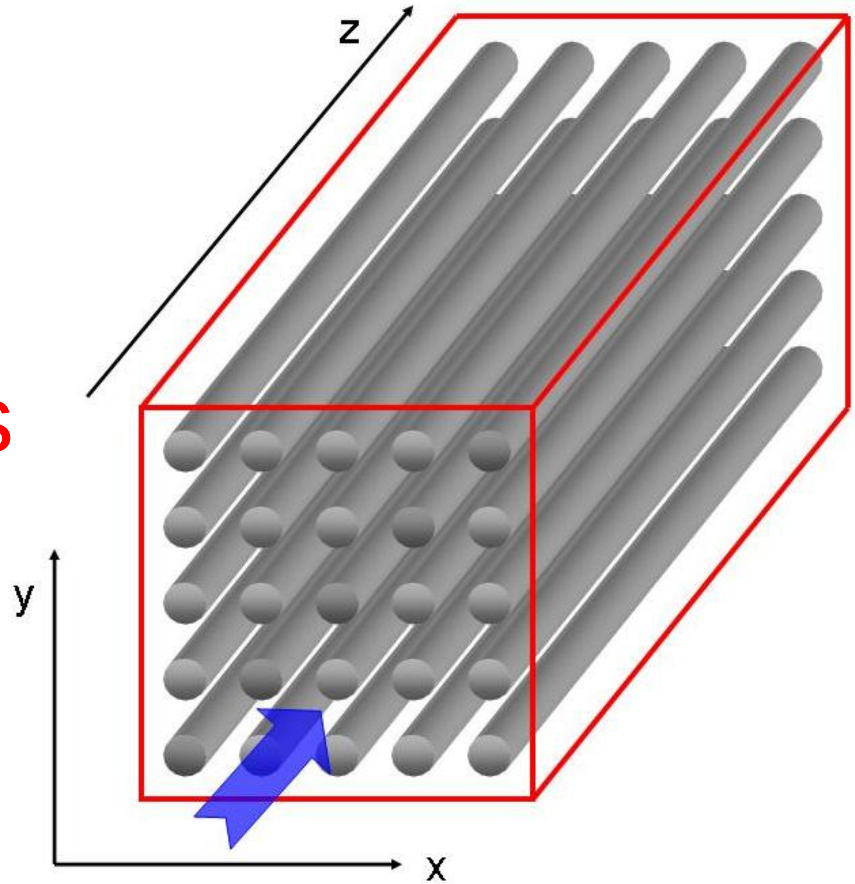
$$\int_{-\infty}^{\infty} \Phi_j \Phi_{j+1} dx \approx 0 \qquad \varepsilon = \int_{-\infty}^{\infty} \left[-a\Phi_j \sin^2 x \Phi_j + \Phi_j \frac{\partial^2 \Phi_j}{\partial x^2} \right] dx$$

$$\mathrm{g} = \int_{-\infty}^{\infty} \left[-a\Phi_j \sin^2 x \Phi_{j+1} + \Phi_j \frac{\partial^2 \Phi_{j+1}}{\partial x^2} \right] dx = \int_{-\infty}^{\infty} \left[-a\Phi_j \sin^2 x \Phi_{j-1} + \Phi_j \frac{\partial^2 \Phi_{j-1}}{\partial x^2} \right] dx$$

$$\psi_j \rightarrow \psi_j \exp(i\varepsilon z)$$

$$i\frac{\partial \psi_j}{\partial z} + g\left(\psi_{j+1} + \psi_{j-1}\right) = 0$$

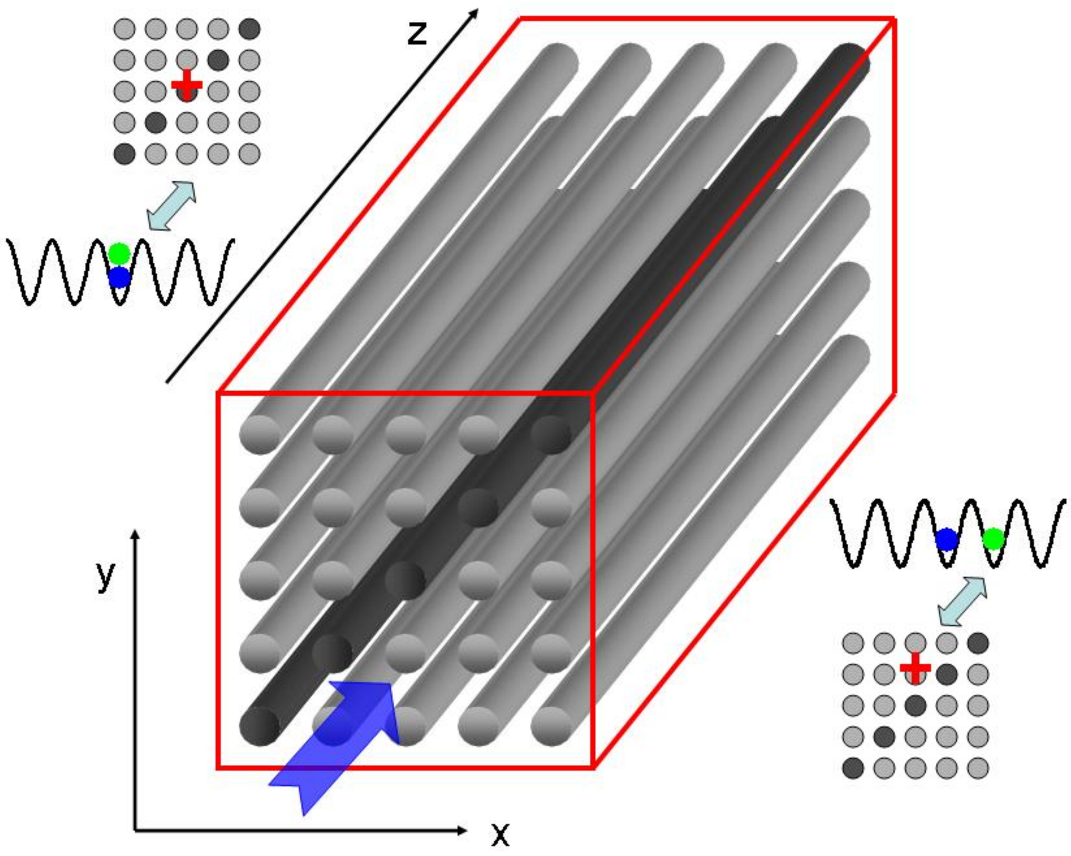
Square Lattice of Optical Waveguides



$$i\frac{\partial}{\partial z}\psi_{mn} = g\left[\psi_{m,n+1} + \psi_{m,n-1} + \psi_{m+1,n} + \psi_{m-1,n}\right]$$

Analogy between Bose Hubbard and Waveguide Lattices

$$\hat{H} = \sum_{j=1}^{N-1}\left(\hat{a}_{j+1}^{+}\hat{a}_{j} + \hat{a}_{j}^{+}\hat{a}_{j+1} + \hat{b}_{j+1}^{+}\hat{b}_{j} + \hat{b}_{j}^{+}\hat{b}_{j+1}\right) + U\sum_{j=1}^{N}\hat{a}_{j}^{+}\hat{b}_{j}^{+}\hat{a}_{j}\hat{b}_{j}$$



$$i\frac{\partial}{\partial z}\psi_{mn} = Q\left[\psi_{m,n+1} + \psi_{m,n-1} + \psi_{m+1,n} + \psi_{m-1,n}\right] + \delta_{mn}U\psi_{mn} + \varepsilon_{mn}\psi_{mn}$$